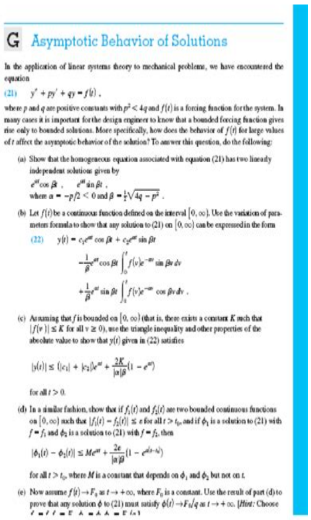

G Asymptotic Behavior of Solutions

In the application of linear systems theory to mechanical problems, we have encountered the equation

(21) $y'' + py' + qy = f(t)$,

where p and q are positive constants with $p^2 < 4q$ and $f(t)$ is a forcing function for the system. In many cases it is important for the design engineer to know that a bounded forcing function gives rise only to bounded solutions. More specifically, how does the behavior of $f(t)$ for large values of t affect the asymptotic behavior of the solution? To answer this question, do the following:

(a) Show that the homogeneous equation associated with equation (21) has two linearly independent solutions given by

$e^{\alpha t}\cos\beta t, \quad e^{\alpha t}\sin\beta t,$

where $\alpha = -p/2 < 0$ and $\beta = \frac{1}{2}\sqrt{4q - p^2}$.

(b) Let $f(t)$ be a continuous function defined on the interval $[0, \infty)$. Use the variation of parameters formula to show that any solution to (21) on $[0, \infty)$ can be expressed in the form

(22) $$y(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t - \frac{1}{\beta}e^{\alpha t}\cos\beta t\int_0^t f(v)e^{-\alpha v}\sin\beta v\,dv + \frac{1}{\beta}e^{\alpha t}\sin\beta t\int_0^t f(v)e^{-\alpha v}\cos\beta v\,dv.$$

(c) Assuming that f is bounded on $[0, \infty)$ (that is, there exists a constant K such that $|f(v)| \le K$ for all $v \ge 0$), use the triangle inequality and other properties of the absolute value to show that $y(t)$ given in (22) satisfies

$$|y(t)| \le (|c_1| + |c_2|)e^{\alpha t} + \frac{2K}{|\alpha|\beta}(1 - e^{\alpha t})$$

for all $t > 0$.

(d) In a similar fashion, show that if $f_1(t)$ and $f_2(t)$ are two bounded continuous functions on $[0, \infty)$ such that $|f_1(t) - f_2(t)| \le \epsilon$ for all $t > t_0$, and if ϕ_1 is a solution to (21) with $f = f_1$ and ϕ_2 is a solution to (21) with $f = f_2$, then

$$|\phi_1(t) - \phi_2(t)| \le Me^{\alpha t} + \frac{2\epsilon}{|\alpha|\beta}(1 - e^{\alpha(t - t_0)})$$

for all $t > t_0$, where M is a constant that depends on ϕ_1 and ϕ_2 but not on t.

(e) Now assume $f(t) \to F_0$ as $t \to +\infty$, where F_0 is a constant. Use the result of part (d) to prove that any solution ϕ to (21) must satisfy $\phi(t) \to F_0/q$ as $t \to +\infty$. [*Hint:* Choose